

Approximability Distance in the Space of H -Colourability Problems

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Abstract. A graph homomorphism is a vertex map which carries edges from a source graph to edges in a target graph. We study the approximability properties of the *Weighted Maximum H -Colourable Subgraph* problem (MAX H -COL). The instances of this problem are edge-weighted graphs G and the objective is to find a subgraph of G that has maximal total edge weight, under the condition that the subgraph has a homomorphism to H ; note that for $H = K_k$ this problem is equivalent to MAX k -CUT. To this end, we introduce a metric structure on the space of graphs which allows us to extend previously known approximability results to larger classes of graphs. Specifically, the approximation algorithms for MAX CUT by Goemans and Williamson and MAX k -CUT by Frieze and Jerrum can be used to yield non-trivial approximation results for MAX H -COL. For a variety of graphs, we show near-optimality results under the Unique Games Conjecture. We also use our method for comparing the performance of Frieze & Jerrum's algorithm with Hastad's approximation algorithm for general MAX 2-CSP. This comparison is, in most cases, favourable to Frieze & Jerrum.

Keywords: optimisation, approximability, graph homomorphism, graph H -colouring, computational complexity

1 Introduction

Let G be a simple, undirected and finite graph. Given a subset $S \subseteq V(G)$, a *cut* in G with respect to S is the edges from a vertex in S to a vertex in $V(G) \setminus S$. The MAX CUT-problem asks for the size of a largest cut in G . More generally, a k -cut in G is the edges going from S_i to S_j , $i \neq j$, where S_1, \dots, S_k is a partitioning of $V(G)$, and the MAX k -CUT-problem asks for the size of a largest k -cut. The problem is readily seen to be identical to finding a largest k -colourable subgraph of G . Furthermore, MAX k -CUT is known to be **APX**-complete for every $k \geq 2$ and consequently does not admit a *polynomial-time approximation scheme* (PTAS).

In the absence of a PTAS, it is interesting to determine the best possible approximation ratio c within which a problem can be approximated or, alternatively the smallest c for which it can be proved that no polynomial-time approximation algorithm exists (typically under some complexity-theoretic assumption such as $\mathbf{P} \neq \mathbf{NP}$). An approximation ratio of .878567 for MAX CUT was obtained in 1995 by Goemans and Williamson [15] using semidefinite programming. Frieze and Jerrum [14] determined lower bounds on the approximation ratios for MAX k -CUT using similar techniques. Sharpened results for small values of k have later been obtained by de Klerk et al. [9]. Under the assumption that the *Unique Games Conjecture* holds, Khot et al. [25] showed the approximation ratio for $k = 2$ to be essentially optimal and also provided upper bounds on the approximation ratio for $k > 2$. Håstad [20] has shown that semidefinite programming is a universal tool for solving the general MAX 2-CSP problem over any domain, in the sense that it establishes non-trivial approximation results for all of those problems.

In this paper, we study approximability properties of a generalised version of MAX k -CUT called MAX H -COL for undirected graphs H . Jonsson et al. [21] have shown that, when H is loop-free, MAX H -COL does not admit a PTAS. Note that if H contains a loop, then MAX H -COL is a trivial problem. We present approximability results for MAX H -COL where H is taken from different families of graphs. Many of these results turn out to be close to optimal under the Unique Games Conjecture. Our approach is based on analysing approximability algorithms applied to problems which they are not originally intended to solve. This vague idea will be clarified below.

Denote by \mathcal{G} the set of all simple, undirected and finite graphs. A *graph homomorphism* h from G to H is a vertex map which carries the edges in G to edges in H . The existence of such a map will be denoted by $G \rightarrow H$. If both $G \rightarrow H$ and $H \rightarrow G$, the graphs G and H are said to be *homomorphically equivalent*. This equivalence will be denoted by $G \equiv H$. For a graph $G \in \mathcal{G}$, let $\mathcal{W}(G)$ be the set of *weight functions* $w : E(G) \rightarrow \mathbb{Q}^+$ assigning weights to edges of G . For a $w \in \mathcal{W}(G)$, we let $\|w\| = \sum_{e \in E(G)} w(e)$ denote the total weight of G . Now, *Weighted Maximum H -Colourable Subgraph* (MAX H -COL) is the maximisation problem with

Instance: An edge-weighted graph (G, w) , where $G \in \mathcal{G}$ and $w \in \mathcal{W}(G)$.

Solution: A subgraph G' of G such that $G' \rightarrow H$.

Measure: The weight of G' with respect to w .

Given an edge-weighted graph (G, w) , denote by $mc_H(G, w)$ the measure of the optimal solution to the problem MAX H -COL. Denote by $mc_k(G, w)$ the (weighted) size of the largest k -cut in (G, w) . This notation is justified by the fact that $mc_k(G, w) = mc_{K_k}(G, w)$. In this sense, MAX H -COL generalises MAX k -CUT. The decision version of MAX H -COL, the *H -colouring* problem has been extensively studied (See [17] and its many references.) and Hell and Nešetřil [16] have shown that the problem is in \mathbf{P} if H contains a loop or is bipartite, and \mathbf{NP} -complete otherwise. Langberg et al. [27] have studied the approximability

of MAX H -COL when H is part of the input. We also note that MAX H -COL is a specialisation of the MAX CSP problem.

The homomorphism relation \rightarrow defines a quasi-order, but not a partial order on the set \mathcal{G} . The failing axiom is that of antisymmetry, since $G \equiv H$ does not necessarily imply $G = H$. To remedy this, let \mathcal{G}_{\equiv} denote the set of equivalence classes of \mathcal{G} under homomorphic equivalence. The relation \rightarrow is defined on \mathcal{G}_{\equiv} in the obvious way and on this set it is a partial order. In fact, \rightarrow provides a lattice structure on \mathcal{G}_{\equiv} and this lattice will be denoted by \mathcal{C}_S . For a more in-depth treatment of graph homomorphisms and the lattice \mathcal{C}_S , see [17]. Here, we endow \mathcal{G}_{\equiv} with a metric d defined in the following way: for $M, N \in \mathcal{G}$, let

$$d(M, N) = 1 - \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_M(G, w)}{mc_N(G, w)} \cdot \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_N(G, w)}{mc_M(G, w)}. \quad (1)$$

In Lemma 5 we will show that d satisfies the following property:

- Let $M, N \in \mathcal{G}$ and assume that mc_M can be approximated within α . Then, mc_N can be approximated within $(1 - d(M, N)) \cdot \alpha$.

Hence, we can use d for extending previously known approximability bounds on MAX H -COL to new and larger classes of graphs. For instance, we can apply Goemans and Williamson’s algorithm (which is intended for solving MAX K_2 -COL) to MAX C_{11} -COL (i.e. the cycle on 11 vertices) and analyse how well the problem is approximated (we will see later on that Goemans and Williamson’s algorithm approximates MAX C_{11} -COL within 0.79869).

In certain cases, the metric d is related to a well-studied graph parameter known as *bipartite density* $b(H)$ [1, 3, 6, 18, 28]: if H' is bipartite subgraph of H with maximum number of edges, then

$$b(H) = \frac{e(H')}{e(H)}.$$

In the end of Section 2 we will see that $b(H) = 1 - d(K_2, H)$ for all edge-transitive graphs H . We note that while d is invariant under homomorphic equivalence, this is not in general true for bipartite density.

The paper is divided into two main parts. Section 2 is used for proving the basic properties of d , showing that it is well-defined on \mathcal{G}_{\equiv} , and that it is a metric. After that, we show that d is computable by linear programming and that the computation of $d(M, N)$ can be simplified whenever M or N is edge-transitive. We conclude this part by providing some examples.

The second part of the paper uses d for studying the approximability of MAX H -COL. For several classes of graphs, we investigate optimality issues by exploiting inapproximability bounds that are consequences of the Unique Games Conjecture. Comparisons are also made to the bounds achieved by the general MAX 2-CSP-algorithm by Håstad [20]. Our investigation covers a spectrum of graphs, ranging from graphs with few edges and/or containing large smallest cycles to graphs containing $\Theta(n^2)$ edges. Dense graphs are considered from two perspectives; firstly as graphs having a number of edges close to maximal and

secondly as graphs from the $\mathcal{G}(n, p)$ model of random graphs, pioneered by Erdős and Rényi [13].

The techniques used in this paper seem to generalise naturally to larger sets of problems. This and other questions are discussed in Section 4 which concludes our paper.

2 Approximation via the Metric d

In this section we start out by proving basic properties of the metric d , that (\mathcal{G}_\equiv, d) is a metric space, and that proximity of graphs M, N in this space lets us interrelate the approximability of MAX M -COL and MAX N -COL. Sections 2.2 and 2.3 are devoted to showing how to compute d .

2.1 The Space (\mathcal{G}_\equiv, d)

We begin by introducing a function $s : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}$ which enables us to express d in a natural way and simplify forthcoming proofs. Let $M, N \in \mathcal{G}$ and define

$$s(M, N) = \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_M(G, w)}{mc_N(G, w)}. \quad (2)$$

The definition of d from (1) can then be written as follows:

$$d(M, N) = 1 - s(N, M) \cdot s(M, N). \quad (3)$$

A consequence of (2) is that the relation $mc_M(G, w) \geq s(M, N) \cdot mc_N(G, w)$ holds for all $G \in \mathcal{G}$ and $w \in \mathcal{W}(G)$. Using this observation, we show that $s(M, N)$ and thereby $d(M, N)$ behaves well under graph homomorphisms and homomorphic equivalence.

Lemma 1. *Let $M, N \in \mathcal{G}$ and $M \rightarrow N$. Then, for every $G \in \mathcal{G}$ and every weight function $w \in \mathcal{W}(G)$,*

$$mc_M(G, w) \leq mc_N(G, w).$$

Proof. If $G' \rightarrow M$ for some subgraph G' of G , then $G' \rightarrow N$ as well. The lemma immediately follows. \square

Corollary 2. *If M and N are homomorphically equivalent, then $mc_M(G, w) = mc_N(G, w)$.*

Corollary 3. *Let $M_1 \equiv M_2$ and $N_1 \equiv N_2$ be two pairs of homomorphically equivalent graphs. Then, for $i, j, k, l \in \{1, 2\}$,*

$$s(N_i, M_j) = s(N_k, M_l).$$

Proof. Corollary 2 shows that for all $G \in \mathcal{G}$ and $w \in \mathcal{W}(G)$, we have

$$\frac{mc_{M_j}(G, w)}{mc_{N_i}(G, w)} = \frac{mc_{M_l}(G, w)}{mc_{N_k}(G, w)}.$$

Now, take the infimum over graphs G and weight functions w on both sides. \square

Corollary 3 shows that s and d are well-defined as functions on the set \mathcal{G}_\equiv . We can now show that d is indeed a metric on this space.

Lemma 4. *The pair (\mathcal{G}_\equiv, d) forms a metric space.*

Proof. Positivity and symmetry follows immediately from the definition and the fact that $s(M, N) \leq 1$ for all M and N . Since $s(M, N) = 1$ if and only if $N \rightarrow M$, it also holds that $d(M, N) = 0$ if and only if M and N are homomorphically equivalent. That is, $d(M, N) = 0$ if and only if M and N represent the same member of \mathcal{G}_\equiv . Furthermore, for graphs M, N and $K \in \mathcal{G}$:

$$\begin{aligned} s(M, N) \cdot s(N, K) &= \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_M(G, w)}{mc_N(G, w)} \cdot \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_N(G, w)}{mc_K(G, w)} \\ &\leq \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_M(G, w)}{mc_N(G, w)} \cdot \frac{mc_N(G, w)}{mc_K(G, w)} = s(M, K). \end{aligned}$$

Therefore, with $a = s(M, N) \cdot s(N, M)$, $b = s(N, K) \cdot s(K, N)$ and $c = s(M, K) \cdot s(K, M) \geq a \cdot b$,

$$\begin{aligned} d(M, N) + d(N, K) - d(M, K) &= 1 - a + 1 - b - (1 - c) \geq \\ &\geq 1 - a - b + a \cdot b = (1 - a) \cdot (1 - b) \geq 0, \end{aligned}$$

which shows that d satisfies the triangle inequality. \square

We say that a maximisation problem Π can be approximated within $c < 1$ if there exists a randomised polynomial-time algorithm A such that $c \cdot \text{Opt}(x) \leq \mathbf{E}(A(x)) \leq \text{Opt}(x)$ for all instances x of Π . Our next result shows that proximity of graphs G and H in d allows us to determine bounds on the approximability of MAX H -COL from known bounds on the approximability of MAX G -COL.

Lemma 5. *Let M, N, K be graphs. If mc_M can be approximated within α , then mc_N can be approximated within $\alpha \cdot (1 - d(M, N))$. If it is **NP**-hard to approximate mc_K within β , then mc_N is not approximable within $\beta / (1 - d(N, K))$ unless $\mathbf{P} = \mathbf{NP}$.*

Proof. Let $A(G, w)$ be the measure of the solution returned by an algorithm which approximates mc_M within α . We know that for all $G \in \mathcal{G}$ and $w \in \mathcal{W}(G)$ we have the inequalities $mc_N(G, w) \geq s(N, M) \cdot mc_M(G, w)$ and $mc_M(G, w) \geq s(M, N) \cdot mc_N(G, w)$. Consequently,

$$\begin{aligned} mc_N(G, w) &\geq mc_M(G, w) \cdot s(N, M) \geq A(G, w) \cdot s(N, M) \\ &\geq mc_M(G, w) \cdot \alpha \cdot s(N, M) \geq mc_N(G, w) \cdot \alpha \cdot s(N, M) \cdot s(M, N) \\ &= mc_N(G, w) \cdot \alpha \cdot (1 - d(M, N)). \end{aligned}$$

For the second part, assume to the contrary that there exists a polynomial-time algorithm B that approximates mc_N within $\beta / (1 - d(N, K))$. According to the first part mc_K can then be approximated within $(1 - d(N, K)) \cdot \beta / (1 - d(N, K)) = \beta$. This is a contradiction unless $\mathbf{P} = \mathbf{NP}$. \square

2.2 Exploiting Symmetries

We have seen that the metric $d(M, N)$ can be defined in terms of $s(M, N)$. In fact, when $M \rightarrow N$ we have $1 - d(M, N) = s(M, N)$. It is therefore of interest to find an expression for s which can be calculated easily. After Lemma 6 (which shows how $mc_M(G, w)$ depends on w) we introduce a different way of describing the solutions to MAX M -COL which makes the proofs of the following results more natural. In Lemma 7, we show that a particular type of weight function provides a lower bound on $mc_M(G, w)/mc_N(G, w)$. Finally, in Lemma 8, we provide a simpler expression for $s(M, N)$ which depends directly on the automorphism group and thereby the symmetries of N . This expression becomes particularly simple when N is edge-transitive. An immediate consequence of this is that $s(K_2, H) = b(H)$ for edge-transitive graphs H .

The optimum $mc_H(G, w)$ is sub-linear with respect to the weight function, as is shown by the following lemma.

Lemma 6. *Let $G, H \in \mathcal{G}$, $\alpha \in \mathbb{Q}^+$ and let $w, w_1, \dots, w_r \in \mathcal{W}(G)$ be weight functions on G . Then,*

$$\begin{aligned} - mc_H(G, \alpha \cdot w) &= \alpha \cdot mc_H(G, w), \\ - mc_H(G, \sum_{i=1}^r w_i) &\leq \sum_{i=1}^r mc_H(G, w_i). \end{aligned}$$

Proof. The first part is trivial. For the second part, let G' be an optimal solution to the instance $(G, \sum_{i=1}^r w_i)$ of MAX H -COL. Then, the measure of this solution equals the sum of the measures of G' as a (possibly suboptimal) solution to each of the instances (G, w_i) . \square

An alternative description of the solutions to MAX H -COL is as follows: let G and $H \in \mathcal{G}$, and for any vertex map $f : V(G) \rightarrow V(H)$, let $f^\# : E(G) \rightarrow E(H)$ be the (partial) edge map induced by f . In this notation $h : V(G) \rightarrow V(H)$ is a graph homomorphism precisely when $(h^\#)^{-1}(E(H)) = E(G)$ or, alternatively when $h^\#$ is a total function. The set of solutions to an instance (G, w) of MAX H -COL can then be taken to be the set of vertex maps $f : V(G) \rightarrow V(H)$ with the measure

$$w(f) = \sum_{e \in (f^\#)^{-1}(E(H))} w(e).$$

In the remaining part of this section, we will use this description of a solution. Let $f : V(G) \rightarrow V(H)$ be an optimal solution to the instance (G, w) of MAX H -COL. Define the weight $w_f \in \mathcal{W}(H)$ as follows: for each $e \in E(H)$, let

$$w_f(e) = \sum_{e' \in (f^\#)^{-1}(e)} \frac{w(e')}{mc_H(G, w)}.$$

We now prove the following result:

Lemma 7. *Let $M, N \in \mathcal{G}$ be two graphs. Then, for every $G \in \mathcal{G}$, every $w \in \mathcal{W}(G)$, and any optimal solution f to (G, w) of MAX N -COL, it holds that*

$$\frac{mc_M(G, w)}{mc_N(G, w)} \geq mc_M(N, w_f).$$

Proof. Arbitrarily choose an optimal solution $g : V(N) \rightarrow V(M)$ to the instance (N, w_f) of MAX M -COL. Then, $g \circ f$ is a solution to (G, w) as an instance of MAX M -COL. The weight of this solution is $mc_M(N, w_f) \cdot mc_N(G, w)$, which implies that

$$mc_M(G, w) \geq mc_M(N, w_f) \cdot mc_N(G, w),$$

and the result follows after division by $mc_N(G, w)$. \square

Let M and $N \in \mathcal{G}$ be graphs and let $A = \text{Aut}(N)$ be the automorphism group of N . We will let $\pi \in A$ act on $\{u, v\} \in E(N)$ by $\pi \cdot \{u, v\} = \{\pi(u), \pi(v)\}$. The graph N is edge-transitive if and only if A acts transitively on the edges of N . Let $\hat{\mathcal{W}}(N)$ be the set of weight functions $w \in \mathcal{W}(N)$ which satisfy $\|w\| = 1$ and for which $w(e) = w(\pi \cdot e)$ for all $e \in E(N)$ and $\pi \in \text{Aut}(N)$.

Lemma 8. *Let $M, N \in \mathcal{G}$. Then,*

$$s(M, N) = \inf_{w \in \hat{\mathcal{W}}(N)} mc_M(N, w).$$

In particular, when N is edge-transitive,

$$s(M, N) = mc_M(N, 1/e(N)).$$

Proof. The easy direction goes through as follows:

$$s(M, N) \leq \inf_{w \in \hat{\mathcal{W}}(N)} \frac{mc_M(N, w)}{mc_N(N, w)} = \inf_{w \in \hat{\mathcal{W}}(N)} mc_M(N, w).$$

For the first part of the lemma, it will be sufficient to prove that the following inequality holds for for some $w' \in \hat{\mathcal{W}}$.

$$\frac{mc_M(G, w)}{mc_N(G, w)} \geq mc_M(N, w') \tag{4}$$

Taking the infimum over graphs G and weight functions $w \in \mathcal{W}(G)$ in the left-hand side of this inequality will then show that

$$s(M, N) \geq mc_M(N, w') \geq \inf_{w \in \hat{\mathcal{W}}(N)} mc_M(N, w).$$

Let $A = \text{Aut}(N)$ be the automorphism group of N . Let $\pi \in A$ be an arbitrary automorphism of N . If f is an optimal solution to (G, w) as an instance of MAX N -COL, then so is $f_\pi = \pi \circ f$. Let $w_\pi = w_{\pi \circ f}$. By Lemma 7, inequality (4) is satisfied by w_π . Summing π in this inequality over A gives

$$|A| \cdot \frac{mc_M(G, w)}{mc_N(G, w)} \geq \sum_{\pi \in A} mc_M(N, w_\pi) \geq mc_M(N, \sum_{\pi \in A} w_\pi),$$

where the last inequality follows from Lemma 6. The weight function $\sum_{\pi \in A} w_\pi$ can be determined as follows.

$$\sum_{\pi \in A} w_\pi(e) = \sum_{\pi \in A} \frac{\sum_{e' \in (f^\#)^{-1}(\pi \cdot e)} w(e')}{mc_N(G, w)} = \frac{|A|}{|Ae|} \cdot \frac{\sum_{e' \in (f^\#)^{-1}(Ae)} w(e')}{mc_N(G, w)},$$

where Ae denotes the orbit of e under A . Thus, $w' \sum_{\pi \in A} w_\pi / |A| \in \hat{\mathcal{W}}(N)$ and w' satisfies (4) so the first part follows.

For the second part, note that when the automorphism group A acts transitively on $E(N)$, there is only one orbit $Ae = E(N)$. Then, the weight function w' is given by

$$w'(e) = \frac{1}{e(N)} \cdot \frac{\sum_{e' \in (f\#)^{-1}(E(N))} w(e')}{mc_N(G, w)} = \frac{1}{e(N)} \cdot \frac{mc_N(G, w)}{mc_N(G, w)}.$$

□

2.3 Tools for Computing Distances

From Lemma 8 it follows that in order to determine $s(M, N)$, it is sufficient to minimise $mc_M(N, w)$ over $\hat{\mathcal{W}}(N)$. We will now use this observation to describe a linear program for computing $s(M, N)$. For $i \in \{1, \dots, r\}$, let A_i be the orbits of $\text{Aut}(N)$ acting on $E(N)$. The measure of a solution f when $w \in \hat{\mathcal{W}}(N)$ is equal to $\sum_{i=1}^r w_i \cdot f_i$, where w_i is the weight of an edge in A_i and f_i is the number of edges in A_i which are mapped to an edge in M by f . Note that given a w , the measure of a solution f depends only on the vector $(f_1, \dots, f_r) \in \mathbb{N}^r$. Therefore, take the solution space to be the set of such vectors:

$$F = \{ (f_1, \dots, f_r) \mid f \text{ is a solution to } (N, w) \text{ of MAX } M\text{-COL} \}$$

Let the variables of the linear program be w_1, \dots, w_r and s , where w_i represents the weight of each element in the orbit A_i and s is an upper bound on the solutions.

$$\begin{aligned} \min s \\ \sum_i f_i \cdot w_i &\leq s \quad \text{for each } (f_1, \dots, f_r) \in F \\ \sum_i |A_i| \cdot w_i &= 1 \\ w_i, s &\geq 0 \end{aligned}$$

Given a solution w_i, s to this program, $w(e) = w_i$ when $e \in A_i$ is a weight function which minimises $mc_M(G, w)$. The value of this solution is $s = s(M, N)$.

Example 9. The *wheel graph* on k vertices, W_k , is a graph that contains a cycle of length $k - 1$ plus a vertex v not in the cycle such that v is connected to every other vertex. We call the edges of the $k - 1$ -cycle *outer edges* and the remaining $k - 1$ edges *spokes*. It is easy to see that W_k contains a maximum clique of size 4 if $k = 4$ (in fact, $W_4 = K_4$) and a maximum clique of size 3 in all other cases. Furthermore, W_k is 3-colourable if and only if k is odd, and 4-colourable otherwise. This implies that for odd k , the wheel graphs are homomorphically equivalent to K_3 .

We will determine $s(K_3, W_n)$ for even $n \geq 6$ using the previously described construction of a linear program. Note that the group action of $\text{Aut}(W_n)$ on $E(W_n)$ has two orbits, one which consists of all outer edges and one which consists of all the spokes. If we remove one outer edge or one spoke from W_k ,

then the resulting graph can be mapped homomorphically onto K_3 . Therefore, it suffices to choose $F = \{f, g\}$ with $f = (k-1, k-2)$ and $g = (k-2, k-1)$ since all other solutions will have a smaller measure than at least one of these. The program for W_k looks like this:

$$\begin{aligned} \min s \\ (k-1) \cdot w_1 + (k-2) \cdot w_2 &\leq s \\ (k-2) \cdot w_1 + (k-1) \cdot w_2 &\leq s \\ (k-1) \cdot w_1 + (k-1) \cdot w_2 &= 1 \\ w_i, s &\geq 0 \end{aligned}$$

The solution is $w_1 = w_2 = 1/(2k-2)$ with $s(K_3, W_k) = s = (2k-3)/(2k-2)$.

Example 10. An example where the weights in the optimal solution to the linear program are different for different orbits is given by $s(K_2, K_{8/3})$. The *rational complete graph* $K_{8/3}$ has vertex set $\{0, 1, \dots, 7\}$, which is thought of as placed on a circle with 0 following 7. There is an edge between any two vertices which are at a distance at least 3 from each other. Each vertex has distance 4 to exactly one other vertex, which means there are 4 such edges. These edges form one orbit A_1 and the remaining 8 edges form the other orbit A_2 . There are two maximal solutions, $f = (0, 8)$ and $g = (4, 6)$ which gives the program:

$$\begin{aligned} \min s \\ 0 \cdot w_1 + 8 \cdot w_2 &\leq s \\ 4 \cdot w_1 + 6 \cdot w_2 &\leq s \\ 4 \cdot w_1 + 8 \cdot w_2 &= 1 \\ w_i, s &\geq 0 \end{aligned}$$

The solution to this program is $w_1 = 1/20$ and $w_2 = 1/10$ with the optimum being $4/5$.

In some cases, it may be hard to determine a desired distance between H and M or N . If we know that H is homomorphically sandwiched between M and N so that $M \rightarrow H \rightarrow N$, then we can provide an upper bound on the distance of H to M or N by using the distance between M and N . Formally, we have:

Lemma 11. *Let $M \rightarrow H \rightarrow N$. Then,*

$$s(M, H) \geq s(M, N) \quad \text{and} \quad s(H, N) \geq s(M, N).$$

Proof. Since $H \rightarrow N$, it follows from Lemma 1 that $mc_H(G, w) \leq mc_N(G, w)$. Thus,

$$s(M, H) = \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_M(G, w)}{mc_H(G, w)} \geq \inf_{\substack{G \in \mathcal{G} \\ w \in \mathcal{W}(G)}} \frac{mc_M(G, w)}{mc_N(G, w)} = s(M, N).$$

The second part follows similarly. □

We will see several applications of this lemma in Sections 3.1 and 3.2.

3 Approximability of MAX H -COL

Let A be an approximation algorithm for MAX H -COL. Our method basically allows us to measure how well A performs on other problems MAX H' -COL. In this section, we will apply the method to various algorithms and various graphs. We do two things for each kind of graph under consideration: compare the performance of our method with that of some existing, leading, approximation algorithm and investigate how close to optimality we can get. Our main algorithmic tools will be the following:

Theorem 12 (Goemans and Williamson [15]). *mc_2 can be approximated within*

$$\alpha_{GW} = \min_{0 < \theta < \pi} \frac{\theta/\pi}{(1 - \cos \theta)/2} \approx .878567.$$

Theorem 13 (Frieze and Jerrum [14]). *mc_k can be approximated within*

$$\alpha_k \sim 1 - \frac{1}{k} + \frac{2 \ln k}{k^2}.$$

Here, the relation \sim indicates two expressions whose ratio tends to 1 as $k \rightarrow \infty$. We note that de Klerk et al. [9] have presented the sharpest known bounds on α_k for small values of k ; for instance, $\alpha_3 \geq 0.836008$.

Let $v(G), e(G)$ denote the number of vertices and edges in G , respectively. Håstad has shown the following:

Theorem 14 (Håstad [20]). *Let H be a graph. There is an absolute constant $c > 0$ such that mc_H can be approximated within*

$$1 - \frac{t(H)}{d^2} \cdot \left(1 - \frac{c}{d^2 \log d}\right)$$

where $d = v(H)$ and $t(H) = d^2 - 2 \cdot e(H)$.

We will compare the performance of this algorithm on MAX H -COL with the performance of the algorithms in Theorems 12 and 13 analysed using Lemma 5 and estimates of the distance d . This comparison is not entirely fair since Håstad's algorithm was probably not designed with the goal of providing optimal results—the goal was to beat random assignments. However, it is the currently best algorithm that can approximate MAX H -COL for arbitrary $H \in \mathcal{G}$. For this purpose, we introduce two functions, FJ_k and $H\hat{a}$, such that, if H is a graph, $FJ_k(H)$ denotes the best bound on the approximation guarantee when Frieze and Jerrum's algorithm for MAX k -CUT is applied to the problem mc_H , while $H\hat{a}(H)$ is the guarantee when Håstad's algorithm is used to approximate mc_H .

To be able to investigate the eventual near-optimality of our approximation method we will rely on the Unique Games Conjecture (UGC). Khot [24] suggested this conjecture as a possible direction for proving inapproximability properties of some important constraint satisfaction problems over two variables. We need the following problem only for stating the conjecture:

Definition 15. *The Unique Label Cover problem $\mathcal{L}(V, W, E, [M], \{\pi^{v,w}\}_{(v,w) \in E}$) is the following problem: Given is a bipartite graph with left side vertices V , right side vertices W , and a set of edges E . The goal is to assign one ‘label’ to every vertex of the graph, where $[M]$ is the set of allowed labels. The labelling is supposed to satisfy certain constraints given by bijective maps $\sigma_{v,w} : [M] \rightarrow [M]$. There is one such map for every edge $(v,w) \in E$. A labelling ‘satisfies’ an edge (v,w) if $\sigma_{v,w}(\text{label}(w)) = \text{label}(v)$. The optimum of the unique label cover problem is defined to be the maximum fraction of edges satisfied by any labelling.*

Now, UGC is the following:

Conjecture 16 (Unique Games Conjecture). For any $\eta, \gamma > 0$, there exists a constant $M = M(\eta, \gamma)$ such that it is **NP**-hard to distinguish whether the Unique Label Cover problem with label set of size M has optimum at least $1 - \eta$ or at most γ .

From hereon we assume that UGC is true, which gives us the following inapproximability results:

Theorem 17 (Khot et al. [25]).

- For every $\varepsilon > 0$, it is NP-hard to approximate mc_2 within $\alpha_{GW} + \varepsilon$.
- It is NP-hard to approximate mc_k within $(1 - 1/k + (2 \ln k)/k^2 + O((\ln \ln k)/k^2))$.

3.1 Sparse Graphs

In this section, we investigate the performance of our method on graphs which have relatively few edges, and we see that the *girth* of the graphs plays a central role. The girth of a graph is the length of a shortest cycle contained in the graph. Similarly, the odd girth of a graph gives the length of a shortest odd cycle in the graph.

Before we proceed we need some facts about cycle graphs. Note that the odd cycles form a chain in the lattice \mathcal{C}_S between K_2 and $C_3 = K_3$ in the following way:

$$K_2 \rightarrow \cdots \rightarrow C_{2i+1} \rightarrow C_{2i-1} \rightarrow \cdots \rightarrow C_3 = K_3.$$

The following lemma gives the values of $s(M, N)$ for pairs of graphs in this chain. The value depends only on the target graph of the homomorphism.

Lemma 18. *Let $k < m$ be positive, odd integers. Then,*

$$s(K_2, C_k) = s(C_m, C_k) = \frac{k-1}{k}.$$

Proof. Note that $C_{2k+1} \not\rightarrow K_2$ and $C_{2k+1} \not\rightarrow C_{2m+1}$. However, after removing one edge from C_{2k+1} , the remaining subgraph is isomorphic to the path P_{2k+1} which in turn is embeddable in both K_2 and C_{2m+1} . Since C_{2k+1} is edge-transitive, the result follows from Lemma 8. \square

With Lemma 18 at hand, we can prove the following:

Proposition 19. *Let $k \geq 3$ be odd. Then, $FJ_2(C_k) \geq \frac{k-1}{k} \cdot \alpha_{GW}$ and $H\hat{a}(C_k) = \frac{2}{k} + \frac{c}{k^2 \log k} - \frac{2c}{k^3 \log k}$. Furthermore, mc_{C_k} cannot be approximated within $\frac{k}{k-1} \cdot \alpha_{GW} + \varepsilon$ for any $\varepsilon > 0$.*

Proof. From Lemma 18 we see that $s(K_2, C_k) = \frac{k-1}{k}$ which implies (using Lemma 5) that $FJ_2(C_k) \geq \frac{k-1}{k} \cdot \alpha_{GW}$. Furthermore, mc_2 cannot be approximated within $\alpha_{GW} + \varepsilon'$ for any $\varepsilon' > 0$. From the second part of Lemma 5, we get that mc_{C_k} cannot be approximated within $\frac{k}{k-1} \cdot (\alpha_{GW} + \varepsilon')$ for any ε' . With $\varepsilon' = \varepsilon \cdot \frac{k-1}{k}$ the result follows.

Finally, we see that

$$\begin{aligned} H\hat{a}(C_k) &= 1 - \frac{k^2 - 2k}{k^2} \cdot \left(1 - \frac{c}{k^2 \log k}\right) = \frac{ck + 2k^2 \log k - 2c}{k^3 \log k} = \\ &= \frac{2}{k} + \frac{c}{k^2 \log k} - \frac{2c}{k^3 \log k}. \end{aligned}$$

Håstad's algorithm does not perform particularly well on sparse graphs; this is reflected by its performance on cycle graphs C_k where the approximation guarantee tends to zero when $k \rightarrow \infty$. We will see that this trend is apparent for all graph types studied in this section.

Now we can continue with a result on a class of graphs with large girth:

Proposition 20. *Let $m > k \geq 4$. If H is a graph with odd girth $g \geq 2k + 1$ and minimum degree $\geq \frac{2m-1}{2(k+1)}$, then $FJ_2(H) \geq \frac{2k}{2k+1} \cdot \alpha_{GW}$ and mc_H cannot be approximated within $\frac{2k+1}{2k} \cdot \alpha_{GW} + \varepsilon$ for any $\varepsilon > 0$. Asymptotically, $H\hat{a}(H)$ is bounded by $\frac{c}{n^2 \log n} + \frac{2(n^{g/(g-1)})^3}{n^4 n^{1/(g-1)}} - \frac{2n^{g/(g-1)} n^{1/(g-1)} c}{n^4 \log n}$, where $n = v(H)$.*

Proof. Lai & Liu [26] have proved that if H is a graph with odd girth $\geq 2k+1$ and minimum degree $\geq \frac{2m-1}{2(k+1)}$, then there exists a homomorphism from H to C_{2k+1} . Thus, $K_2 \rightarrow H \rightarrow C_{2k+1}$ which implies that $1 - d(K_2, H) \geq 1 - d(K_2, C_{2k+1}) = \frac{2k}{2k+1}$. By Lemma 5, $FJ_2(H) \geq \frac{2k}{2k+1} \cdot \alpha_{GW}$, but mc_H cannot be approximated within $\frac{2k+1}{2k} \cdot \alpha_{GW} + \varepsilon$ for any $\varepsilon > 0$.

Dutton and Brigham [10] show that one upper bound on $e(H)$ has asymptotic order $n^{1+2/(g-1)}$. This lets us say that

$$\begin{aligned} H\hat{a}(H) &\sim 1 - \frac{n^2 - 2 \cdot n^{1+2/(g-1)}}{n^2} \cdot \left(1 - \frac{c}{n^2 \log n}\right) = \\ &= \frac{cn^2 + 2n^{(3g-1)/(g-1)} \log n - 2n^{(g+1)/(g-1)} c}{n^4 \log n} = \\ &= \frac{c}{n^2 \log n} + \frac{2(n^{g/(g-1)})^3}{n^4 n^{1/(g-1)}} - \frac{2n^{g/(g-1)} n^{1/(g-1)} c}{n^4 \log n}. \end{aligned}$$

□

If we restrict ourselves to planar graphs, then it is possible to show the following:

Proposition 21. *Let H be a planar graph with girth at least $g = \frac{20k-2}{3}$. If $v(H) = n$, then $FJ_2(H) \geq \frac{2k}{2k+1} \cdot \alpha_{GW}$ and $H\hat{a}(H) \leq \frac{6}{n} - \frac{12}{n^2} + \frac{c}{n^2 \log n} - \frac{6c}{n^3 \log n} + \frac{12c}{n^4 \log n} \cdot mc_H$ cannot be approximated within $\frac{2k+1}{2k} \cdot \alpha_{GW} + \varepsilon$ for any $\varepsilon > 0$.*

Proof. Borodin et al. [7] have proved that H is $(2 + \frac{1}{k})$ -colourable which is equivalent to saying that there exists a homomorphism from H to C_{2k+1} . The proof proceeds as for Proposition 20.

The planar graph H cannot have more than $3n-6$ edges so $H\hat{a}(H)$ is bounded from above by

$$\begin{aligned} 1 - \frac{n^2 - 2(3n-6)}{n^2} \cdot \left(1 - \frac{c}{n^2 \log n}\right) &= \\ &= \frac{cn^2 - 6nc + 12c + 6n^3 \log n - 12n^2 \log n}{n^4 \log n} = \\ &= \frac{6}{n} - \frac{12}{n^2} + \frac{c}{n^2 \log n} - \frac{6c}{n^3 \log n} + \frac{12c}{n^4 \log n}. \end{aligned}$$

(In fact, H contains no more than $\max\{g(n-2)/(g-2), n-1\}$ edges, but using this only makes for a more convoluted expression to study.) \square

Proposition 21 can be strengthened and extended in different ways: one is to consider a result by Dvořák et al. [11]. They have proved that every planar graph H of odd-girth at least 9 is homomorphic to the Petersen graph P . The Petersen graph is edge-transitive and it is known (cf. [3]) that the bipartite density of P is $4/5$ or, in other words, $s(K_2, P) = 4/5$. Consequently, mc_H can be approximated within $\frac{4}{5} \cdot \alpha_{GW}$ but not within $\frac{4}{5} \cdot \alpha_{GW} + \varepsilon$ for any $\varepsilon > 0$. This is better than Proposition 21 for planar graphs with girth strictly less than 13.

Another way of extending Proposition 21 is to consider graphs embeddable on higher-genus surfaces. For instance, the lemma is true for graphs embeddable on the projective plane, and it is also true for graphs of girth *strictly* greater than $\frac{20k-2}{3}$ whenever the graphs are embeddable on the torus or Klein bottle. These bounds are direct consequences of results in Borodin et al.

We conclude the section by looking at a class of graphs that have small girth. Let $0 < \beta < 1$, be the approximation threshold for mc_3 , i.e. mc_3 is approximable within β but not within $\beta + \varepsilon$ for any $\varepsilon > 0$. Currently, we know that $\alpha_3 \leq 0.836008 \leq \beta \leq \frac{102}{103}$ [9, 22]. The wheel graphs W_k from Example 9 are homomorphically equivalent to K_3 for odd k and we conclude (by Lemma 5) that mc_{W_k} has the same approximability properties as mc_3 in this case. For even $k \geq 6$, W_k is not homomorphically equivalent to K_3 , though.

Proposition 22. *For $k \geq 6$ and even, $FJ_3(W_k) \geq \alpha_3 \cdot \frac{2k-3}{2k-2}$ but mc_{W_k} is not approximable within $\beta \cdot \frac{2k-2}{2k-3}$. $H\hat{a}(W_k) = \frac{4}{k} - \frac{4}{k^2} + \frac{c}{k^2 \log k} - \frac{4c}{k^3 \log k} + \frac{4c}{k^4 \log k}$.*

Proof. We know from Example 9 that $K_3 \rightarrow W_k$ and $s(K_3, W_k) = \frac{2k-3}{2k-2}$. The first part of the result follows by an application of Lemma 5.

$$H\hat{a}(W_k) = 1 - \frac{t(W_k)}{d^2} \cdot \left(1 - \frac{c}{d^2 \log d}\right) = /d = k, e(W_k) = 2(k-1)/ =$$

$$\begin{aligned}
&= 1 - \frac{k^2 - 4(k-1)}{k^2} \cdot \left(1 - \frac{c}{k^2 \log k}\right) = \\
&= \frac{k^2 c + 4k^3 \log k - 4kc - 4k^2 \log k + 4c}{k^4 \log k} = \\
&= \frac{4}{k} - \frac{4}{k^2} + \frac{c}{k^2 \log k} - \frac{4c}{k^3 \log k} + \frac{4c}{k^4 \log k}
\end{aligned}$$

□

We see that $FJ_3(W_k) \rightarrow \alpha_3$ when $k \rightarrow \infty$, while $H\hat{a}(W_k)$ tends to 0.

3.2 Dense and Random Graphs

We will now study *dense* graphs, i.e. graphs H containing $\Theta(v(H)^2)$ edges. For a graph H on n vertices, we obviously have $H \rightarrow K_n$. If we assume that $\omega(H) \geq r$, then we also have $K_r \rightarrow H$. Thus, if we could determine $s(K_r, K_n)$, then we could use Lemma 11 to calculate a bound on $FJ_n(H)$.

Let $\omega(G)$ denote the size of the largest clique in G and $\chi(G)$ denote the chromatic number of G . The Turán graph $T(n, r)$ is a graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices whenever they belong to different subsets. Turán graphs have the following properties [31]:

- $e(T(n, r)) = \lfloor (1 - \frac{1}{r}) \cdot \frac{n^2}{2} \rfloor$;
- $\omega(T(n, r)) = \chi(T(n, r)) = r$;
- if G is a graph such that $e(G) > e(T(n, r))$, then $\omega(G) > r$.

Lemma 23. *Let r and n be positive integers. Then,*

$$s(K_r, K_n) = e(T(n, r))/e(K_n)$$

Proof. Since K_n is edge-transitive, it suffices to show that $mc_r(K_n, 1/e(K_n)) = e(T(n, r))/e(K_n)$. Assume $mc_r(K_n, 1/e(K_n)) \cdot e(K_n) > e(T(n, r))$. This implies that there exists an r -partite graph G on n vertices with strictly more than $e(T(n, r))$ edges — this is impossible since $\omega(G) > r$ and, consequently, $\chi(G) > r$. Thus, $mc_{K_r}(K_n, 1/e(K_n)) \cdot e(K_n) = e(T(n, r))$ because $T(n, r)$ is an r -partite subgraph of K_n . □

Now, we are ready to prove the following:

Proposition 24. *Let $v(H) = n$ and pick $r \in \mathbb{N}$, $\sigma \in \mathbb{R}$ such that*

$$\left\lfloor \left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2} \right\rfloor \leq \sigma \cdot n^2 = e(H) \leq \frac{n(n-1)}{2}.$$

Then,

$$FJ_n(H) \geq \alpha_n \cdot \frac{2 \left\lfloor \left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2} \right\rfloor}{n \cdot (n-1)} \sim 1 - \frac{1}{r} - \frac{1}{n} + \frac{2 \ln n}{n(n-1)}$$

$$H\hat{a}(H) = 2\sigma + \frac{c}{n^2 \log n} - \frac{2\sigma \cdot c}{n^2 \log n}.$$

Proof. We have $K_r \rightarrow H$ due to Turán and $H \rightarrow K_n$ holds trivially since $v(H) = n$. By Lemma 23

$$s(K_r, K_n) = \frac{2 \left\lfloor \left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2} \right\rfloor}{n \cdot (n-1)}.$$

The first part of the result follows from Lemma 5 since $d(H, K_n) \leq d(K_r, K_n) = 1 - s(K_r, K_n)$ and some straightforward calculations.

$$\begin{aligned} H\hat{a}(H) &= 1 - \frac{n^2 - \sigma \cdot n^2}{n^2} \cdot \left(1 - \frac{c}{n^2 \log n}\right) = \\ &= \frac{c + 2\sigma \cdot n^2 \log n - 2\sigma \cdot c}{n^2 \log n} = \frac{c}{n^2 \log n} + 2\sigma - \frac{2\sigma \cdot c}{n^2 \log n}. \end{aligned}$$

□

Note that when r and n grow, $FJ_n(H)$ tends to 1. This means that, asymptotically, we cannot do much better. If we compare the expression for $FJ_n(H)$ with the inapproximability bound for mc_n (Theorem 17), we see that all we could hope for is a faster convergence towards 1. As σ satisfies $(1 - \frac{1}{r}) \cdot \frac{1}{2} \leq \sigma \leq (1 - \frac{1}{n}) \cdot \frac{1}{2}$, we conclude that $H\hat{a}(H)$ also tends to 1 as r and n grow. To get a better grip on how $H\hat{a}(H)$ behaves we look at two extreme cases.

For a maximal $\sigma = (1 - \frac{1}{r}) \cdot \frac{1}{2}$, $H\hat{a}(H)$ becomes

$$1 - \frac{1}{n} + \frac{c}{n^3 \log n}.$$

On the other hand, this guarantee, for a minimal $\sigma = (1 - \frac{1}{r}) \cdot \frac{1}{2}$ is

$$1 - \frac{1}{r} + \frac{c}{rn^2 \log n}.$$

At the same time, it is easy to see that Frieze and Jerrum's algorithm makes these points approximable within α_n (since, in this case, $H \equiv K_n$) and α_r (since Turán's theorem tells us that $H \rightarrow K_r$ holds in this case), respectively. Our conclusion is that Frieze and Jerrum's and Håstad's algorithms perform almost equally well on these graphs asymptotically.

Another way to study dense graphs is via random graphs. Let $\mathcal{G}(n, p)$ denote the random graph on n vertices in which every edge is chosen randomly and independently with probability $p = p(n)$. We say that $\mathcal{G}(n, p)$ has a property A *asymptotically almost surely* (a.a.s.) if the probability it satisfies A tends to 1 as n tends to infinity. Here, we let $p = c$ for some $0 < c < 1$.

For $G \in \mathcal{G}(n, p)$ it is well known that a.a.s. $\omega(G)$ assumes one of at most two values around $\frac{2 \ln n}{\ln(1/p)}$ [5, 30]. It is also known that, almost surely $\chi(G) \sim \frac{n}{2 \ln(np)} \ln \left(\frac{1}{1-p} \right)$, as $np \rightarrow \infty$ [4, 29]. Let us say that $\chi(G)$ is concentrated in width s if there exists $u = u(n, p)$ such that a.a.s. $u \leq \chi(G) \leq u + s$. Alon and Krivelevich [2] have shown that for every constant $\delta > 0$, if $p = n^{-1/2-\delta}$ then $\chi(G)$ is concentrated in width $s = 1$. That is, almost surely, the chromatic number takes one of two values.

Proposition 25. Let $H \in \mathcal{G}(n, p)$. When $np \rightarrow \infty$, $FJ_m(H) \sim 1 - \frac{2}{m} + \frac{2 \ln m}{m^2} + \frac{1}{m^2} - \frac{2 \ln m}{m^3}$, where $m = \omega(H)$. $H\hat{a}(H) = p - \frac{p}{n} + (1-p) \cdot \frac{c}{n^2 \log n} + \frac{pc}{n^3 \log n}$.

Proof. Let $k = \chi(H)$.

$$\begin{aligned} FJ_m(H) &\geq \alpha_m \cdot s(K_m, K_k) \sim \left(1 - \frac{1}{m} + \frac{2 \ln m}{m^2}\right) \cdot \frac{2 \left\lfloor \left(1 - \frac{1}{m}\right) \cdot \frac{k^2}{2} \right\rfloor}{k(k-1)} \sim \\ &\sim \frac{(m^2 - m + 2 \ln m)(m-1)k}{m^3(k-1)} = \\ &= \frac{k}{k-1} - \frac{2k}{m(k-1)} + \frac{k}{m^2(k-1)} + \frac{2k \ln m}{m^2(k-1)} - \frac{2k \ln m}{m^3(k-1)} \quad (**) \end{aligned}$$

If n is large, then $k \gg m$ and

$$(**) \sim 1 - \frac{2}{m} + \frac{2 \ln m}{m^2} + \frac{1}{m^2} - \frac{2 \ln m}{m^3}.$$

The expected number of edges for a graph $H \in \mathcal{G}(n, p)$ is $\binom{n}{2}p$, so

$$\begin{aligned} H\hat{a}(H) &= 1 - \frac{t(G)}{d^2} \cdot \left(1 - \frac{c}{d^2 \log d}\right) = /d = n, e(G) = \binom{n}{2}p / = \\ &= 1 - \frac{n^2 - (n^2 - n)p}{n^2} \cdot \left(1 - \frac{c}{n^2 \log n}\right) = 1 - \frac{n - pn + p}{n} \cdot \left(1 - \frac{c}{n^2 \log n}\right) = \\ &= 1 - \left(1 - p + \frac{p}{n}\right) \cdot \left(1 - \frac{c}{n^2 \log n}\right) = \\ &= \frac{pn^3 \log n + nc - pnc - pn^2 \log n + pc}{n^3 \log n} = \\ &= p - \frac{p}{n} + (1-p) \cdot \frac{c}{n^2 \log n} + \frac{pc}{n^3 \log n} \end{aligned}$$

□

We see that, in the limiting case, $H\hat{a}(H)$ tends to p , while $FJ_m(H)$ tends to 1. Again, this means that, for large enough graphs, we cannot do much better. With a better analysis, one could possibly reach an expression for $FJ_m(H)$ that has a faster convergence rate.

Of course, it is interesting to look at what happens for graphs $H \in \mathcal{G}(n, p)$ where np does not tend to ∞ when $n \rightarrow \infty$. The following theorem lets us do this.

Theorem 26 (Erdős and Rényi [13]). Let c be a positive constant and $p = \frac{c}{n}$. If $c < 1$, then a.a.s. no component in $\mathcal{G}(n, p)$ contains more than one cycle, and no component has more than $\frac{\ln n}{c-1-\ln c}$ vertices.

Now we see that if $np \rightarrow \varepsilon$ when $n \rightarrow \infty$ and $0 < \varepsilon < 1$, then $\mathcal{G}(n, p)$ almost surely consists of components with at most one cycle. Thus, each component resembles a cycle where, possibly, trees are attached to certain cycle vertices, and each component is homomorphically equivalent to the cycle it contains. Since we know from Section 3.1 that Frieze and Jerrum's algorithm performs better than Håstad's algorithm on cycle graphs, it follows that the same relationship holds in this part of the $\mathcal{G}(n, p)$ spectrum.

4 Conclusions and Open Problems

We have seen that applying Frieze and Jerrum’s algorithm to MAX H -COL gives comparable to or better results than when applying Håstad’s MAX 2-CSP algorithm for the classes of graphs we have considered. One possible explanation for this is that the analysis of the MAX 2-CSP algorithm only aims to prove it better than a random solution on expectation, which may leave room for strengthenings of the approximation guarantee. At the same time, we are probably overestimating the distance between the graphs. It is likely that both results can be improved.

Kaporis et al. [23] have shown that mc_2 is approximable within .952 for any given average degree d and asymptotically almost all random graphs G in $\mathcal{G}(n, m = \lfloor \frac{d}{2}n \rfloor)$, where $\mathcal{G}(n, m)$ is the probability space of random graphs on n vertices and m edges selected uniformly at random. In a similar vein, Coja-Oghlan et al. [8] give an algorithm that approximates mc_k within $1 - O(1/\sqrt{np})$ in expected polynomial time, for graphs from $\mathcal{G}(n, p)$. It would be interesting to know if these results could be carried further, to other graphs G , so that better approximability bounds on MAX H -COL, for H such that $G \rightarrow H$, could be achieved.

Erdős [12] has proved that for any positive integers k and l there exists a graph of chromatic number k and girth at least l . It is obvious that such graphs cannot be sandwiched between K_2 and a cycle as was the case of the graphs of high girth in Section 3.1. A different idea is thus required to deal with these graphs. In general, to apply our method more precisely, we need a better understanding of the structure of \mathcal{C}_S and how this interacts with our metric d .

The idea of defining a metric on a space of problems which relates their approximability can be extended to more general cases. It should not prove too difficult to generalise the framework introduced in this paper to MAX CSP over directed graphs or even languages consisting of a single, finitary relation. How far can this generalisation be carried out? Could it provide any insight into the approximability of MAX CSP on arbitrary constraint languages?

When considering inapproximability, we have strongly relied on the Unique Games Conjecture—hence, we are part of the growing body interested in seeing UGC settled. We note, though, that weaker inapproximability results exist for both MAX CUT [19] and MAX k -CUT [22] and that they are applicable in our setting. We want to emphasise that our method is not *per se* dependent on the truth of the UGC.

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